

State of matter at high density and entropy bounds

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Entropy of all systems that we understand well is proportional to their volumes except for black holes given by their horizon area. This makes the microstates of any quantum theory of gravity drastically different from the ordinary matter. Because of the assumption that black holes are the maximum entropy states there have been many conjectures that put the area, defined one way or another, as a bound on the entropy in a given region of spacetime. Here we construct a simple model with entropy proportional to volume which exceeds the entropy of a single black hole. We show that a homogeneous cosmology filled with this gas exceeds one of the tightest entropy bounds, the covariant entropy bound and discuss the implications.

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Bekenstein's discovery [1] of black hole entropy and Hawking's calculation [2] of the temperature are among the most important clues we have on a quantum theory of gravity. These show a significant difference between the microstates of quantum gravity and ordinary matter. Quantum field theory, the way we understand it, implies that the density of states in momentum space is proportional to the volume and so is the entropy.

The assumption, or the bias, that black holes are the highest entropy states brought about many conjectures on the maximum entropy confined in a region. In the absence of gravity, one can put arbitrary entropy in a region by inserting enough energy in it. Gravity changes this picture. Energy distorts the geometry and volume in such a way that the entropy is no longer arbitrarily large. Bekenstein [3] conjectured that in a system where gravity is not the dominant interaction, the entropy in a region is bounded by the entropy of a black hole of the same size. This conjecture fails in well known examples. Consider a cosmology with flat spatial slices and ordinary matter, like radiation. The entropy is proportional to the volume which grows much faster than the area and exceeds it for large enough regions. Later, several more refined versions of this conjecture were stated [4, 5] that survive in more general circumstances.

All of these conjectures assume some constraint on the energy-momentum tensor like the weak or dominant energy conditions. It is not possible to put any bound on entropy if one allows an arbitrary energy-momentum tensor. For example, consider a combination of positive and negative energy states with a zero net energy-momentum. These states would not have any feedback on geometry and hence, can violate any entropy bound. Therefore, in a sense, the entropy bounds are limitations on the form of energy-momentum tensor rather than restricting the entropy. Putting enough restrictions on the energy-momentum tensor, one can prove some of the bounds in weak gravity limit [6, 7]. However, we should be cautious about putting too much constraint on the states of a theory (quantum gravity) that we do not understand well.

Here we motivate a heuristic model which satisfies the dominant energy condition and breaks one of the tightest entropy bounds, the covariant (Bousso) bound [5]. The details and the subtleties of this model is discussed in [8, 9].

Let's investigate the maximum entropy one can put in a cube of size L . As depicted in Fig.1, at small energy densities we expect a gas of relativistic particles to have the highest entropy. Increasing the energy in the box makes it collapse into a black hole which indeed has a much higher entropy. Can we put more entropy and energy in this box? The answer is yes and a simple configuration is motivated in Fig.2. Each of the 27 smaller holes on the right panel have $1/9$ of the area and entropy of the hole on the left. Hence, the net entropy of the right panel is 3 times

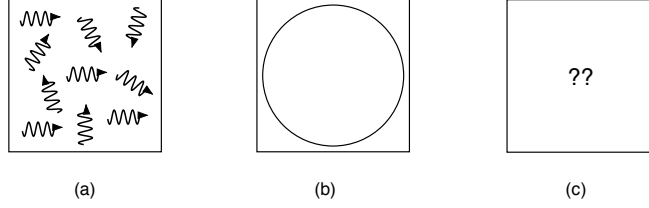


FIG. 1: For small densities we expect the maximum entropy state to be a gas of relativistic particles (left), increasing the energy we expect the matter to collapse into a black hole (middle). Does a phase with higher energy and entropy exist(right)?

of the single hole configuration.

Using N^3 holes of size L/N , we get N times more entropy than a single black hole. If this box is put in empty space, it is entropically favorable for the smaller ones to merge into a single black hole. The result is a black hole of size N^2L and N^5 times the entropy of the original lattice. However, if the whole universe is filled with this black hole lattice, there is no room for them to merge. It is entropically favorable to have this lattice until the universe expands enough and they can merge then.

We may worry that this system is completely unstable to gravitational collapse. However, to surpass the single black hole entropy we do not necessarily create strong gravitational fields. One can build a gas of small black holes where each is well outside the Schwarzschild radius of the others. Suppose the separation between the black holes is n times larger than their Schwarzschild radius. To exceed the entropy of the single black hole configuration, we need small black holes of size $R \leq L/n^3$. Consider a solar system size box in which the black holes are separated by 10000 times of their horizon radius. We will need small black holes of size 5 meters to surpass the single black hole entropy. The big separation of the black holes prevents an immediate coalescence. This example shows that even using the simple ingredients of classical gravity we can get entropies larger than the one for a single black hole.

In more realistic scenarios obtained from string theory we replace this lattice of black holes with a set of intersecting branes. These branes interact with each other and have pressure and thermodynamics. Let's consider a gas of these black holes with typical sizes R_s and separation of the same order R_s . Each black hole has entropy $S_{\text{bh}} \sim R_s^2/G$ and energy $E_{\text{bh}} \sim R_s/G$ and there are $N_{\text{bh}} \sim V/R_s^3$ of them in a volume V . The total energy and entropy is given by

$$S \sim N_{\text{bh}} S_{\text{bh}} \sim \frac{V}{R_s G}, \quad E \sim N_{\text{bh}} E_{\text{bh}} \sim \frac{V}{R_s^2 G}. \quad (1)$$

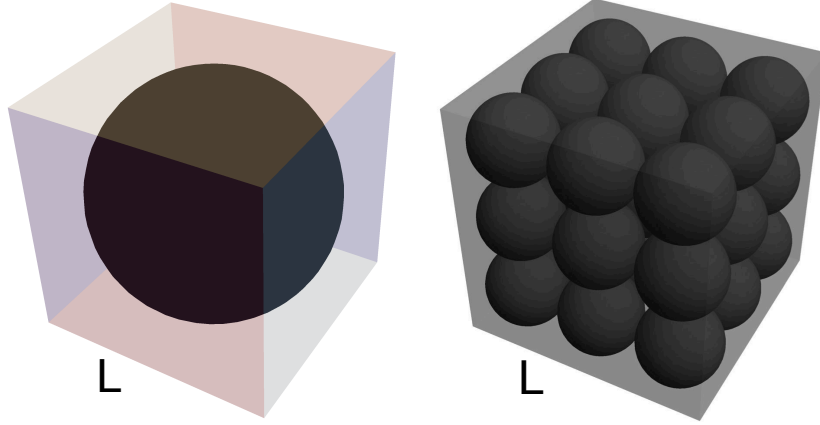


FIG. 2: Left, a box of size L contains a single black hole. Right, the same box, filled with 27 smaller black holes of size $L/3$. The combined horizon area of smaller black holes is 3 times larger than the left panel.

Eliminating R_s between these two equations we get

$$S \sim \frac{1}{R_s} \frac{V}{G} \sim \left(\frac{V}{EG} \right)^{-\frac{1}{2}} \frac{V}{G} \sim \sqrt{\frac{EV}{G}} \quad (2)$$

Introducing a constant K of order unity and rewriting in terms of energy and entropy densities ρ and s we get

$$s = K \sqrt{\frac{\rho}{G}}. \quad (3)$$

This gives an extensive equation for entropy. Using the first law of thermodynamics $dE = TdS - pdV$ we get

$$\begin{aligned} T &= \left(\frac{\partial S}{\partial E} \right)_V^{-1} = \frac{2}{K} \sqrt{\frac{EG}{V}}, \\ p &= T \left(\frac{\partial S}{\partial V} \right)_E = \frac{E}{V} = \rho. \end{aligned} \quad (4)$$

This corresponds to the equation of state $p = \rho$ which satisfies the dominant energy condition. Although we derived Eq.(3) using a heuristic model, this equation has been derived from many different approaches including:

1. Veneziano [10] obtained it assuming that the entropy is given by the area of the apparent horizon with radius $H^{-1} \sim (G\rho)^{-\frac{1}{2}}$.
2. Banks and Fishler [11] derived $p = \rho$ using coalescing horizon size black holes.

3. Horowitz and Polchinski [12] derived it for a gas of string states near the point where the strings are large enough to collapse into a black hole.
4. Sasakura [13] conjectured it by demanding a ‘spacetime uncertainty relation’.
5. Verlinde [14] argued that a similar relation correspond to the Cardy formula for the density of states.
6. Brustein and Veneziano [15] obtained it by using the notion of a causal connection.
7. Masoumi and Mathur [8] showed that Eq.(3) is the only entropy formula which is manifestly invariant under S and T-dualities and approaches the black hole entropy at small densities.

Now we show that this high-entropy ‘matter’ can violate one of the most beautiful entropy bounds, the covariant (Bousso) bound. This bound is based on the entropy in the light sheet and has a very different philosophy and formulation. The question of interest in the earlier bounds was: How much is the maximum entropy one can put in a given region? The answers were typically the area of that region in Planck units. The covariant entropy bound changes the question: Given a two-dimensional surface \mathcal{S} of area A , what is the region which has a maximum entropy given by A ? The conjecture was expressed in terms of non-expanding null geodesic congruences. Take a bundle of null rays perpendicular to \mathcal{S} and follow them until they reach a caustic or start diverging. The region covered by this bundle is called the light-sheet. The conjecture expresses that the entropy of this light-sheet is bounded by A . We show that a homogeneous cosmology with entropy density given in Eq.(3) exceeds this bound. Assume a Bianchi type I metric

$$ds^2 = -dt^2 + \sum_{i=1}^3 a_i(t)^2 dx_i^2 . \quad (5)$$

This is a homogenous anisotropic universe. The Friedmann equations for this universe filled with the matter discussed in Eq.(3) have a power law solution in the form

$$a_i(t) = a_{0i} t^{C_i} \text{ for } 0 < C_i < 1 ., \quad (6)$$

where $\sum_{i=1}^3 C_i = 1$. Let’s choose a surface \mathcal{S} in the $(x_2 - x_3)$ plane. If S_{bound} is the maximum entropy allowed by the covariant bound, we can show

$$\text{ratio} = \frac{S_{\text{LightSheet}}}{S_{\text{bound}}} = \frac{K (1 - \sum_i C_i^2)^{\frac{1}{2}}}{\sqrt{\pi}(1 - C_1)} . \quad (7)$$

This ratio can get bigger than one for *any* value of K if C_1 is close enough to one. For example if $K = 1$ we need $C_1 = 0.85$ which is a mild anisotropy. The ratio remains the same even if we do not

follow the light sheet all the way to the initial singularity. This is in contrast with the conditions for ordinary matter like radiation. To break the bound for radiation we must follow the light sheet to Planckian densities where we have no good understanding.

Here we constructed a model which violates the tightest of entropy bounds. There is no simple reason to assume this model is not allowed on first principles. If a shear viscosity prevents large anisotropy we may be able to find a deeper connection between the viscosity and entropy bounds. If this violation stands as it is, it has important consequences. The simplest would be tightening the assumptions of the area-based entropy bounds. This would render these conjectures less useful. As mentioned above, the entropy bounds are really constraints on the type of matter allowed by the theory. The bounds would not be very impressive under too much limitation. A deeper consequence would be returning to the more natural bounds based on the volume instead of area. This will bring the number counting for the states allowed by gravity to the same footing as other quantum theories where we have a much better understanding and control.

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